THE STATIONARY PRINCIPLE OF COMPLEMENTARY WORK IN NONLINEAR THEORY OF ELASTICITY

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It is shown that Euler's equations and the natural boundary conditions of the variational problem of the stationary state of complementary work are the equations of continuity written in components of the Piola stress tensor and the boundary conditions on that part of the surface where the displacements are given.

The complementary work is regarded as a functional of the Piola stress tensor. The static equations for the Piola stress tensor are written in the metric of the initial (undeformed) state. This approach permits the separation of the static and the geometrical sides of the problem of equilibrium of an elastic body. For an isotropic elastic medium a method is shown for the expression of complementary work through the components of the Piola tensor. The basic notation with respect to nonlinear theory of elasticity is taken from [1].

The variation of specific potential strain energy for an ideal elastic body is [1]

$$\delta W = \frac{1}{2}\left(\frac{G}{g}\right)^{1/2} \mathbf{Q} \cdot\cdot\, \delta \mathbf{G}^* = \frac{1}{2}\left(\frac{G}{g}\right)^{1/2} I_1\{\mathbf{Q}\cdot\delta\mathbf{G}^*\} \tag{1}$$

$$(G^* = \nabla R \cdot \nabla R^T)$$

Here $\mathbf{Q}$ is the stress energy tensor [1], G^* is Cauchy's measure of strain, $\mathbf{R}$ is the radius vector of a point of the deformed body, ∇ is the nabla operator in the metric of the undeformed state, $I_1\{P\}$ is the first invariant of the tensor $\mathbf{P}$, and $G\,/\,g$ is the third invariant of the tensor G^*.

We take advantage of the following equations:

$$\sqrt{G/g}\,\mathbf{Q} = \mathbf{D}\cdot(\nabla\mathbf{R})^{-1}, \qquad \delta\mathbf{G}^* = \nabla\mathbf{R}\cdot\nabla\delta\mathbf{R}^T + \nabla\delta\mathbf{R}\cdot\nabla\mathbf{R}^T$$

where $\mathbf{D}$ is the Piola stress tensor [1], and express (1) in the form

$$\delta W = {}^1/_2 I_1\{\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}\cdot\nabla\mathbf{R}\cdot\delta\nabla\mathbf{R}^T\} + {}^1/_2\,[\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}]\cdot\cdot[\nabla\delta\mathbf{R}\cdot\nabla\mathbf{R}^T]$$

Since tensor $\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}$ is symmetric, we have

$$[\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}]\cdot\cdot[\nabla\delta\mathbf{R}\cdot\nabla\mathbf{R}^T] = [\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}]\cdot\cdot[\nabla\delta\mathbf{R}\cdot\nabla\mathbf{R}^T]^T =$$

$$= [\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}]\cdot\cdot[\nabla\mathbf{R}\cdot\nabla\delta\mathbf{R}^T] = I_1\{\mathbf{D}\cdot(\nabla\mathbf{R})^{-1}\cdot\nabla\mathbf{R}\cdot\nabla\delta\mathbf{R}^T\}$$

In this manner we obtain instead of (1)

$$\delta W = \mathbf{D}\cdot\cdot\nabla\delta\mathbf{R}^T \tag{2}$$

If we write $\nabla\mathbf{R} = \mathbf{C}$ and consider W as a function of components of tensor $\mathbf{C}$, then it follows from (2) that

$$\partial_{sk} = \partial W\,/\,\partial C^{sk}$$

Here ∂_{sk} and C^{sk} are components of tensors $\mathbf{D}$ and $\mathbf{C}$ with some basis.

We bring into consideration the specific complementary strain work as a function of components of tensor $\mathbf{D}$. This function is related to W through the Legendre transformation

$$B = \mathbf{D}\cdot\cdot\mathbf{C}^T - W \tag{3}$$

According to the property of Legendre transformation [2]

$$\delta B = C^T \cdot\cdot \delta D = \nabla R^T \cdot\cdot \delta D$$

Let the elastic body in its undeformed state occupy the volume v bounded by the surface $o = o_1 + o_2$; the external surface forces are given on o_1, while the displacements are given on o_2.

It is known from [1] that equilibrium equations in the volume and on the surface of an elastic body can be written in the form

$$\nabla \cdot D + \rho_0 \mathbf{K} = 0 \quad \text{in } v, \qquad \mathbf{n} \cdot D = \mathbf{F}^\circ \quad \text{on } o_1 \tag{4}$$

Here ρ_0 is the density of the medium in the undeformed state, $\mathbf{K}$ is the mass force vector, $\mathbf{n}$ is the normal to the surface of the undeformed body, $\mathbf{F}^\circ$ is the vector of external surface forces per unit area of the undeformed body.

The arbitrary tensor D which satisfies conditions (4) will be called statically possible.

Let us examine the following functional of statically possible tensors D which is referred to as complementary work:

$$\Phi = \iiint_v B \, d\tau - \iint_{o_2} \mathbf{R} \cdot \mathbf{F}^\circ \, do \tag{5}$$

Here $\mathbf{F}^\circ$ should be treated as the reaction to adjustments which ensure the equality of the displacement vector on o_2 to its prescribed value.

Further, referring to (2) and taking into account that on o_1 vector $\mathbf{R}$ does not vary, we obtain

$$\delta\Phi = \iiint_v \nabla \mathbf{R}^T \cdot\cdot \delta D \, d\tau - \iint_{o_2} \mathbf{R} \cdot \delta \mathbf{F}^\circ \, do \tag{6}$$

Assuming that vector $\mathbf{R}$ is continuously differentiable, we integrate (6) by parts with the aid of the following identity:

$$P^T \cdot\cdot \nabla \mathbf{a} = \nabla \cdot (P \cdot \mathbf{a}) - (\nabla \cdot P) \cdot \mathbf{a} \tag{7}$$

We arrive at the equation

$$\delta\Phi = -\iiint_v (\nabla \cdot \delta D) \cdot \mathbf{R} \, d\tau + \iint_{o_1} \mathbf{n} \cdot \delta D \cdot \mathbf{R} \, do + \iint_{o_2} \mathbf{n} \cdot \delta D \cdot \mathbf{R} \, do - \iint_{o_2} \mathbf{R} \cdot \delta \mathbf{F}^\circ \, do = 0$$

because according to (4)

$$\nabla \cdot \delta D = 0 \quad \text{in } v, \qquad \mathbf{n} \cdot \delta D = 0 \quad \text{on } o_1, \qquad \mathbf{n} \cdot \delta D = \delta \mathbf{F}^\circ \quad \text{on } o_2 \tag{8}$$

In this manner the stationary state of complementary work follows from the continuity of the medium.

Now we shall show the inverse: the continuity equations

$$\nabla \times C = 0 \quad \text{in } v$$

and the boundary conditions on o_2 turn out to be Euler's equations and the natural boundary conditions of the variational problem of the stationary state of complementary work.

Since variations δD must satisfy the condition (8), we introduce (*) the Lagrangian vector $\boldsymbol{\lambda}$

$$\delta\Phi = \iiint_v [C^T \cdot\cdot \delta D + \boldsymbol{\lambda} \cdot (\nabla \cdot \delta D)] \, d\tau - \iint_{o_2} \mathbf{R} \cdot \delta \mathbf{F}^\circ \, do$$

Now through the appropriate choice of vector $\boldsymbol{\lambda}$ we can consider the variations of

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*) This method of proof was recommended to the author by Lur'e.

components of tensor D as independent. Further, referring to (7) we have

$$(\nabla\cdot\delta\mathrm{D})\cdot\boldsymbol{\lambda} = \nabla\cdot(\delta\mathrm{D}\cdot\boldsymbol{\lambda}) - \delta\mathrm{D}\cdot\cdot\nabla\boldsymbol{\lambda}^T$$

The variational equation assumes the form

$$\delta\Phi = \iiint_v (\mathrm{C}^T - \nabla\boldsymbol{\lambda}^T)\cdot\cdot\delta\mathrm{D}\,d\tau + \iint_{o_2} (\boldsymbol{\lambda} - \mathbf{R})\cdot\delta\mathbf{F}^\circ do = 0$$

From the arbitrariness of the variation $\delta\mathrm{D}$ in the volume and on o_2 we obtain that the tensor C must be equal to the gradient of some vector; this is equivalent to the equation

$$\nabla \times \mathrm{C} = 0 \tag{9}$$

This vector itself computed from the tensor C is equal to the given vector $\mathbf{R}$ on the surface o_2. Since the tensor C here is assumed to be expressed in terms of the tensor D, Eq. (9) is the condition of continuity expressed in terms of the components of the Piola stress tensor.

Now it is necessary to solve the problem of expressing the tensor $\nabla\mathbf{R}$ and the complementary work Φ in terms of the Piola stress tensor. For an isotropic elastic medium the following method may be proposed.

Again we take advantage of equation

$$\mathrm{D} = \sqrt{G/g}\,\mathrm{Q}\cdot\nabla\mathbf{R}$$

and form the symmetric tensor

$$\mathrm{D}\cdot\mathrm{D}^T = (G/g)\,\mathrm{Q}\cdot\mathrm{G}^*\cdot\mathrm{Q}$$

For an isotropic elastic medium the tensors Q and G^* are coaxial. Consequently, the tensor $\mathrm{D}\cdot\mathrm{D}^T$ is coaxial with the tensor G^* and can be represented in the form

$$\mathrm{D}\cdot\mathrm{D}^T = a\mathrm{E} + b\mathrm{G}^* + c\mathrm{G}^{*2} \tag{10}$$

where a, b and c are functions of the invariants of tensor G^*. Relationship (10) can be transformed, i.e. we can write

$$\mathrm{G}^* = a_1\mathrm{E} + b_1\mathrm{D}\cdot\mathrm{D}^T + c_1(\mathrm{D}\cdot\mathrm{D}^T)^2 \tag{11}$$

Here a_1, b_1 and c_1 are functions of invariants of tensor $\mathrm{D}\cdot\mathrm{D}^T$. By analogy we can write

$$\sqrt{G/g}\,\mathrm{Q} = a_2\mathrm{E} + b_2\mathrm{D}\cdot\mathrm{D}^T + c_2(\mathrm{D}\cdot\mathrm{D}^T)^2 \tag{12}$$

where a_2, b_2 and c_2 are also some functions of invariants of tensor $\mathrm{D}\cdot\mathrm{D}^T$.

For an isotropic medium the specific potential strain energy is a function of invariants of Cauchy's measure of strain G^*. From relationship (11) we can express the invariants of tensor G^* in terms of the invariants of the tensor $\mathrm{D}\cdot\mathrm{D}^T$. By the same token the specific potential strain energy W will be expressed in terms of the components of the Piola stress tensor. Furthermore, the quantity

$$\mathrm{D}\cdot\cdot\nabla\mathbf{R}^T = \sqrt{G/g}\,\mathrm{Q}\cdot\cdot\mathrm{G}^*$$

is also expressed in terms of the invariants of tensor $\mathrm{D}\cdot\mathrm{D}^T$ with the aid of Eqs. (11) and (12). In this manner the specific complementary strain work B of an isotropic elastic body turns out to be represented as a function of invariants of tensor $\mathrm{D}\cdot\mathrm{D}^T$, i.e. as a final result it is represented as a function of components of the Piola stress tensor. It follows from (12) that the tensor $\sqrt{g/G}\,\mathrm{Q}^{-1}$ is also an isotropic tensor function of tensor $\mathrm{D}\cdot\mathrm{D}^T$ and can be represented in the form

$$\sqrt{g/G}\,\mathrm{Q}^{-1} = a_3\mathrm{E} + b_3\mathrm{D}\cdot\mathrm{D}^T + c_3(\mathrm{D}\cdot\mathrm{D}^T)^2$$

Here a_3, b_3 and c_3 are again functions of invariants of $D \cdot D^T$. Therefore, when the tensor D is known, the tensor $\nabla \mathbf{R}$ is determined by the relationship

$$\nabla \mathbf{R} = [a_3 E + b_3 D \cdot D^T + c_3 (D \cdot D^T)^2] \cdot D$$

The radius vector of the deformed body is determined by equation

$$\mathbf{R} = \int_{(M_0)}^{(M)} d\mathbf{r} \cdot [a_3 E + b_3 D \cdot D^T + c_3 (D \cdot D^T)^2] \cdot D + \mathbf{R}(M_0)$$

where the integral can be computed on any curve connecting the points M_0 and M.

Let us perform the outlined computation using the example of a semilinear material [1]. For a semilinear material the expression for the specific potential strain energy W and the equation of state have the following form:

$$W = {}^1/_2 \lambda s_1^2 + \mu s_2 = {}^1/_2 D \cdot\cdot (\nabla \mathbf{R}^T - A^T), \quad A = G^{*-1/2} \cdot \nabla \mathbf{R} \qquad (13)$$

$$D = [(\lambda s_1 - 2\mu) G^{*-1/2} + 2\mu E] \cdot \nabla R \qquad (14)$$

$$s_1 = I_1\{G^{*1/2}\} - 3, \quad s_2 = I_1\{G^*\} - 2I_1\{G^{*1/2}\} + 3 \qquad (\lambda, \mu = \mathrm{const})$$

According to Eq. (14) we obtain

$$D \cdot D^T = [(\lambda s_1 - 2\mu) E + 2\mu G^{*1/2}]^2$$

hence

$$2\mu G^{*1/2} = (D \cdot D^T)^{1/2} - (\lambda s_1 - 2\mu) E, \quad s_1 = f_1 / (3\lambda + 2\mu), \quad f_1 = I_1\{(D \cdot D^T)^{1/2}\}$$

Furthermore,

$$[(\lambda s_1 - 2\mu) G^{*-1/2} + 2\mu E]^{-1} = G^{*1/2} \cdot (D \cdot D^T)^{-1/2} =$$
$$= \frac{1}{2\mu}\left[E - \left(\frac{\lambda f_1}{3\lambda + 2\mu} - 2\mu\right)(D \cdot D^T)^{-1/2}\right]$$

Now we obtain from (14)

$$\nabla \mathbf{R} = \frac{1}{2\mu}\left[E - \left(\frac{\lambda f_1}{3\lambda + 2\mu} - 2\mu\right)(D \cdot D^T)^{-1/2}\right] \cdot D \qquad (15)$$

We note that relationship (15) remains essentially nonlinear with respect to tensor D no matter how small the stresses.

We compute further

$$D \cdot\cdot \nabla \mathbf{R}^T = \frac{1}{2\mu}\left[D \cdot\cdot D^T - \left(\frac{\lambda f_1}{3\lambda + 2\mu} - 2\mu\right) f_1\right], \qquad D \cdot\cdot A^T = f_1$$

Finally, according to (13) we obtain the expression for the specific complementary strain work for the semilinear material in terms of the first invariants of tensors $(D \cdot D^T)^{1/2}$ and $D \cdot D^T$

$$B = {}^1/_2 D \cdot\cdot \nabla \mathbf{R}^T + {}^1/_2 D \cdot\cdot A^T = \frac{1}{4\mu}\left[D \cdot\cdot D^T - \frac{\nu}{1+\nu} f_1^2\right] + f_1, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

Note. The representation of specific complementary strain work in the form of (3) appears in monograph [3] with the unessential difference that instead of the radius vector **R** of the deformed body the vector of displacements is used.

However, in [3] the specific complementary strain work is considered as a function of both the Piola stress tensor and the gradients of displacements; the assertion is made in this connection that it is impossible to express the complementary work in terms of the components of the Piola stress tensor only.

If the complementary work is regarded as a functional of both the Piola stress tensor and the vector of displacements, the meaning of Castigliano's principle is lost as a variational principle, which selects among all statically possible states of stress those which satisfy the conditions of continuity.

In paper [4] the complementary work is treated as a functional of the Piola stress tensor only and it is established that continuity equations (9) follow from the stationary state of complementary work. Nevertheless, the question about the possibility of expressing the gradients of displacements and the specific complementary strain work in terms of components of the Piola stress tensor remains open in paper [4].

In fact, as was shown above, the gradients of displacements and the specific complementary strain work can be represented as a function of the components of the Piola stress tensor only. Therefore the principle of Castigliano which was formulated for the Piola stress tensor retains its significance also in the nonlinear theory of elasticity.

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BIBLIOGRAPHY

1. Lur'e, A. I., Theory of elasticity for a semilinear material. PMM Vol. 32, №6, 1968.
2. Lur'e, A. I., Analytical Mechanics, M., Fizmatgiz, 1961.
3. Novozhilov, V. V., Theory of Elasticity. L., Sudpromgiz, 1958.
4. Levinson, M., The complementary energy theorem in finite elasticity. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 87, p. 826, 1965.

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ON PLANE CONTACT PROBLEMS OF THE THEORY OF ELASTICITY IN THE PRESENCE OF ADHESION OR FRICTION

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As is known, plane contact problems of the theory of elasticity for a half-plane in the presence of adhesion or friction in the contact domain have been studied sufficiently well.

Corresponding contact problems for elastic solids different in shape or their mechanical properties from an isotropic elastic half-plane, have begun to be worked upon comparatively recently. The papers of Popov [1, 2] should here be singled out first.

A general analysis of the structure of the solution of nonclassical plane contact problems in the presence of adhesion or friction in the contact domain is given herein. Possible methods of solving them effectively are indicated.

1. Mathematical formulation. Some auxiliary results. We call the following the nonclassical mixed problems: (1) mixed problems of elasticity theory for bodies of complex shape (strip, layer, circle, sphere, infinite cylinder, wedge, etc.),